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Efficient Solution of Boolean Equations Using Variable-Entered Karnaugh Maps

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ABSTRACT. A new method for obtaining a compact subsumptive general solution of a system of Boolean equations is presented. The method relies on the use of the variable-entered Karnaugh map (VEKM) to achieve successive elimination through successive map folding. It also makes an artificial distinction between don't-care and can't-happen conditions. Therefore, it is highly efficient as it requires the construction of maps that are both *significantly fewer and significantly smaller* than those required by classical methods. Moreover, the method is applicable to general Boolean equations and is not restricted to the two-valued case. Details of the method are formally justified, carefully explained and further demonstrated via an illustrative example.

1. Introduction

The topic of Boolean equations has been a hot topic of research for almost two centuries and its importance can hardly be overestimated. Boolean-equation solving permeates many areas of modern science such as logical design, biology, grammars, chemistry, law, medicine, spectroscopy, and graph theory [1]. Many important problems in operations research can be reduced to the problem of solving a system of Boolean equations. A notable example is the problem of an n-person coalition game with a domination relation between different strategies [2]. The solutions of Boolean equations serve also as an important tool in the treatment of pseudo-Boolean equations and inequalities, and their associated problems in integer linear programming [2].

The key step used in solving Boolean equations is that of deriving eliminants: entities that can be readily represented on maps. Brown [1] considered the possibility of using a Conventional Karnaugh Map (CKM) in which the rows and columns are arranged according to a reflected binary code, or using a Marquand diagram (also called Veitch chart) in which natural binary order is used. Though CKMs are easier to construct and read than Marquand diagrams, Brown [1] chose to employ Marquand diagrams to solve Boolean equations since the rules for using them are easier to state than those for using CKMs.

Tucker and Tapia [3.4] developed a new CKM method for solving two-valued Boolean equations. Their method makes a clever distinction between don't-care and can't-happen conditions, and hence requires *significantly fewer maps* than a typical CKM method does. Though this method is presented without proof, some intuitive understanding of the reason for its mappings is given. Similarly to Brown [1], Tucker and Tapia [3,4] stated their rules in cellwise tabular form.

Rushdi [5] developed yet another mapping method for obtaining a subsumptive general solution of a system of Boolean equations. This method is not restricted to the two-valued case and requires the construction of *maps* that are *significantly smaller* than those required by existing procedures. This is because it relies on the use of a more powerful map, namely the variable-entered Karnaugh map (VEKM). The VEKM is an adaptation of the CKM that retains most of its pictorial insight and effectively combines algebraic and mapping techniques. Historically, the VEKM was developed to double the variable-handling capability of the CKM [6]. Later, the VEKM was shown to be the direct or natural map for finite Boolean algebras other than the bivalent or 2-valued Boolean algebra (switching algebra) [1,5,7,8]. These algebras are sometimes called 'big' Boolean algebras, and are useful and unavoidable, even if unrecognizable, in many applications [1].

In the present work, we propose a combination of the method of Tucker and Tapia [3,4] and that of Rushdi [5], i.e., we develop a mapping method that: (a) distinguishes don't-care and can't-happen conditions and (b) employs the VEKM to obtain a subsumptive general solution of a system of Boolean equations. The proposed combined method is highly efficient, as it requires the construction of maps that are both *significantly fewer and significantly smaller* than those required by classical methods. The rules of using these maps are easy to remember, as they are stated in a collective algebraic form. The algebraic rules have an inherent nice regularity, and hence are preferable to the earlier cellwise tabular ones that obscure such regularities. As an offshoot of the present work, a formal proof of the basic concepts in Tucker and Tapia [3,4] is given.

The organization of the rest of this short note is as follows. Section 2 reviews the concept of subsumptive general solutions of a Boolean system of equations, while section 3 presents a simplified reformulation of the constructions in section 2. This reformulation leads to efficient constructions of subsumptive solutions that can use genuine VEKM representations or can use a CKM or an algebraic representation, which are the two opposite degenerate VEKM representations. An illustrative example follows in section 4 which serves not only to demonstrate the method proposed herein, but also to show its superiority to earlier methods in simplicity, efficiency, and compactness of solution form. Section 5 is devoted to some concluding remarks.

2. Subsumptive General Solutions

This section is a review of the classical technique of constructing subsumptive general solutions for a system of Boolean equations [1,5,9]. We consider an n-variable Boolean system on a Boolean algebra **B** that is usually a set of k, $k \geq 1$, simultaneously asserted equations and/or inequalities. Such a system is always reducible [1] to an equivalent single equation

$$f(\mathbf{X}) = 0, \tag{2.1}$$

where $\mathbf{X} = [X_1, X_2, \ldots, X_n]^T$ is a vector of n components X_i, each belonging to the Boolean carrier **B**. The symbol f in (2.1) denotes an n-variable Boolean function $f : \mathbf{B}^n \rightarrow \mathbf{B}$, whose eliminants $f_n(X_1, X_2, \ldots, X_n), \ldots, f_i(X_1, X_2, \ldots, X_{i-1}, X_i), \ldots, f_2(X_1, X_2), f_1(X_1), f_0$ are constructed by setting $f_n = f$ and using the recursion

$$f_{i-1}(X_1, X_2, \ldots, X_{i-1}) = (f_i / \overline{X}_i) \wedge (f_i / X_i), \quad i = n, n-1, \ldots, 1. \tag{2.2}$$

The function f_{i-1}, called the conjunctive eliminant of f_i with respect to the singleton $\{X_i\}$, is a conjunction of the two ratios or subfunctions

$$f_i / \overline{X}_i = f_i(X_1, X_2, \ldots, X_{i-1}, 0), \tag{2.3a}$$

$$f_i / X_i = f_i(X_1, X_2, \ldots, X_{i-1}, 1), \tag{2.3b}$$

obtained from f_i by setting or restricting X_i in it to 0 and to 1, respectively. These two ratios will be denoted by f_i (0) and f_i(1), respectively. A subsumptive general solution is produced by successive elimination of variables, a technique transforming the problem (2.1) of solving a single equation of n variables to that of solving n equations of one variable each. The solution requires a separate consistency condition

$$f_0 = 0 , \tag{2.4a}$$

plus expressing each of the pertinent variables as an interval of functions of the preceding variables, namely, for i = 1, 2, …,n :

$$s_i (X_1 , X_2, ..., X_{i-1}) \leq X_i \leq t_i (X_1, X_2, \dots, X_{i-1}), \tag{2.4b}$$

where the s_i and t_i functions are expressed as incompletely specified Boolean functions (ISBFs) again in the interval form [1]

$$f_i (0)\ \overline{f_i} (1) \leq s_i \leq f_i (0), \tag{2.5a}$$

$$\overline{f_i} (1) \leq t_i \leq f_i (0) \vee \overline{f_i} (1). \tag{2.5b}$$

The form of the general solution above allows all the particular solutions of (2.1), and nothing else, to be generated as a tree. Such a generation is of a modest computational complexity, and is performed only whenever needed. In fact, spatial economy dictates that a solution be displayed in a compact general form rather than as a list of particular solutions.

3. A Simplified Reformulation

In this section we reformulate the subsumptive general solution (2.4)-(2.5) into a new form suitable for efficient computation. We note that the subsumptive solution has two sources of specification incompleteness: (a) each variable X_i is given as an interval of functions and (b) the bounding functions s_i and t_i are also given in interval form. The interval form for X_i in (2.4b) can be rewritten in the don't-care form [5]

$$X_i = s_i \vee d (\overline{s_i}\ t_i). \tag{3.1}$$

Similarly, the interval forms for s_i and t_i in (2.5) are rewritten in the don't-care forms

$$s_i = f_i (0)\ \overline{f_i} (1) \vee c (f_i (0)\ f_i (1)), \tag{3.2a}$$

$$t_i = \quad \overline{f}_i(1) \quad \vee \quad c\,(\, f_i(0)\;\; f_i(1)\,), \tag{3.2b}$$

where the new type of don’t cares in (3.2) is labeled as "c" rather than "d" so as to be distinguished from the earlier type in (3.1). The symbol "c" has been particularly chosen so as to denote a can’t-happen condition, which is a special case of don’t-care conditions. We will temporarily make an artificial distinction between can’t-happen conditions and ordinary don’t-care conditions. In this, we conform to the terminology of Tucker and Tapia [3,4] who made such a distinction in their table/map work. The substitution of (3.2) into (3.1) results in the following expression for X_i

$$X_i = f_i(0)\;\; \overline{f}_i(1) \vee c(\, f_i(0)\;\; f_i(1)\,) \vee d\,(\; \overline{f}_i(0)\;\; \overline{f}_i(1)\,). \tag{3.3}$$

Note that (3.3) expresses X_i as an ORing of three *disjoint* parts: the (positively) asserted part $f_i(0)\ \overline{f}_i(1)$, the can’t-happen part $c\ (\ f_i\ (0)\ f_i\ (1)\ )$ and the don’t-care part $d(\ \overline{f}_i\ (0)\ \ \overline{f}_i\ (1)\ )$. We stress that (3.3) is valid for a general Boolean algebra and is not restricted to the 2-valued case. However, to illustrate our results and demonstrate their correspondence with these of Tucker and Tapia [3,4], we specialize (3.3) to the 2-valued Boolean algebra in Table 1 which summarizes the values that X_i takes for all possible combinations of binary values for $f_i(0)$ and $f_i(1)$.

Table 1: The value of X_i in terms of those of $f_i(0)$ and $f_i(1)$ for the 2-valued case.

$f_i(0)$	$f_i(1)$	X_i : Meaning
1	1	c can’t happen
1	0	1 (positively) asserted
0	1	0 (negatively) asserted
0	0	d don’t care

It is possible to argue that the three parts in (3.3) need not really be disjoint since the asserted part can "creep" into the c and d parts [Note that $1 \vee c = 1$, $1 \vee d = 1$], and that (3.3) can be rewritten in the simplified form

$$X_i = f_i(0)\;\; \overline{f}_i(1) \quad \vee \quad c\,(\, f_i(0)\,) \quad \vee \; d\,(\; \overline{f}_i(1)\,). \tag{3.4}$$

However, we will refrain from doing so at this stage as we will deliberately maintain representations for the variables X_i, $i = n, n\text{-}1, \ldots, 2, 1$, in the disjoint form (3.3) and insist on our artificial distinction between the c and d conditions therein. Such an X_i representation contains all the information necessary for the solution to proceed at its *i*th iteration, viz., it contains full information on the functions s_i, t_i and f_{i-1}. The two bounding functions s_i and t_i are easily recovered from (3.3) as follows. In view of (3.2a), the lower bound s_i can be obtained from the right hand side of (3.3) by setting d to 0. Moreover, the distinction between c and d is not needed in the resulting expression of s_i, and hence c is switched into d in that expression to yield

$$\begin{aligned} s_i &= f_i(0)\ \overline{f}_i(1) \ \vee\ d\,(f_i(0)\ f_i(1)) \\ &= f_i(0)\ \overline{f}_i(1) \ \vee\ d\,(f_i(0)), \end{aligned} \tag{3.5}$$

while the upper bound t_i is obtained from the right hand side of (3.3) by setting d to 1 and again switching c into d, viz.,

$$\begin{aligned} t_i &= f_i(0)\ \overline{f}_i(1) \ \vee\ d\,(f_i(0)\ f_i(1)) \ \vee\ \overline{f}_i(0)\ \overline{f}_i(1) \\ &= \overline{f}_i(1) \ \vee\ d\,(f_i(0)). \end{aligned} \tag{3.6}$$

Finally, we observe that the can't-happen part in (3.3), i.e. $f_i(0)\ f_i(1)$ is simply the next eliminant f_{i-1}. This can't-happen part can be expanded into its two subfunctions $f_{i-1}(0)$ and $f_{i-1}(1)$ which are all we need to construct a representation for the next variable X_{i-1}. We can proceed recursively to generate each successor variable X_{i-1} from its predecessor X_i for i = n-1, n-2, …, 1. To construct a representation for the starting variable X_n, we need the eliminant $f_n = f$, which can be generated from a fictitious variable X_{n+1} in the form

$$X_{n+1} = c\,(\,f\,(\mathbf{X})\,). \tag{3.7}$$

Note that while the representation (3.3) involves three ANDing (logical multiplication) operations, we need only to implement two of them. We multiply $f_i(0)$ and $f_i(1)$ to form the c part $f_i(0)\ f_i(1)$ needed for f_{i-1} and also multiply $f_i(0)$ and $\overline{f}_i(1)$ since this is the asserted part of s_i in (3.5). However, we neither need to multiply $\overline{f}_i(0)$ and $f_i(1)$ nor to construct $\overline{f}_i(0)$ in the first place. The product $\overline{f}_i(0)\ \overline{f}_i(1)$ is not needed for s_i, t_i or f_{i-1}.

So far, we have considered a general representation for the X_i's that can be either purely algebraic, purely map (e.g., a CKM) or a mixture thereof (i.e., a VEKM). Note that the natural pure map representation has the largest maps among all VEKM representations while a purely algebraic representation has the most complicated manipulations among all VEKM representations. Note also that the natural pure map for a 'big' Boolean algebra has entries that include constants other than the usual constants 0 and 1 of the 2-valued Boolean algebra. Such a map can be called a CKM on its pertinent algebra, but is easier to handle as a VEKM on the 2-valued Boolean algebra with the extra constants (the ones other than 1 and 0) viewed as functions of entered "variables" [1,5,7,8]. This means that while we have several choices for the 2-valued Boolean algebra, we do not have much of a choice when handling 'big' Boolean algebras in which case VEKMs seem to possess a clear advantage.

If the pertinent Boolean algebra is 2-valued and the X_i representation is a CKM, then the method of Tucker and Tapia [3,4] is recovered as a special case of the current method. In this case, the map of a predecessor variable X_{i+1} acts as a generator map for the map of a successor variable X_i. The can't-happen (c) entries in the generator or X_{i+1} map represent the cases when f_{i+1} (0) f_{i+1} (1) = f_i is asserted. The generator X_{i+1} map is split into two halves: $X_i = 0$ (which is a map for the subfunction $f_i(0)$ defining it as 1 for c entries and 0 otherwise) and $X_i = 1$ (which is a map for the subfunction $f_i(1)$ again defining it as 1 for c entries and 0 otherwise). Now, the X_i map is generated from the X_{i+1} map by eliminating the map variable X_i from the X_{i+1} map and combining every two cells in the X_{i+1} map that share common values for all map variables other than X_i itself. Such two cells for $X_i = 0$ and for $X_i = 1$ combine to give a single cell in the X_i map. The entry in the combined cell depends on the entries in the original cells, which can be either a 'c', or not. Note that when an entry is not a 'c', it is immaterial whether it is a '1', a '0' or a 'd', and can collectively be denoted as a '*'. A 'c' ('*') entry in the X_i = 0 cell means that f_i (0) is 1(0). Similarly, a 'c' ('*') entry in the X_i = 1 cell means that f_i (1) is 1(0). These latter statements can be used to translate Table 1 into Table 2 which decides the value of a cell in the X_i map in terms of the values of the corresponding two cells in the X_{i+1} or generator map. A table similar to Table 2 has earlier appeared in Tucker and Tapia [3] but it was based only on some intuitive understanding.

Table 2: The value of a cell in the X_i map in terms of the values of the corresponding two cells in the generator map for the 2-valued case.

Value for the $X_i = 0$ cell in the X_{i+1} map	Value for the $X_i = 1$ cell in the X_{i+1} map	Value in the corresponding cell in the X_i map
c	c	c
c	*	1
*	c	0
*	*	d

Now, if the pertinent Boolean algebra is not restricted, and the sequence of variables X_i has VEKM representations, it is straightforward to use the VEKM of a predecessor variable X_{i+1} as a generator map for the next variable X_i. The rule of converting the X_{i+1} VEKM into the X_i VEKM depends on whether X_i is a map variable or an entered variable for the X_{i+1} VEKM and is shown in Fig. 1, which is simply a graphical translation of (3.3).

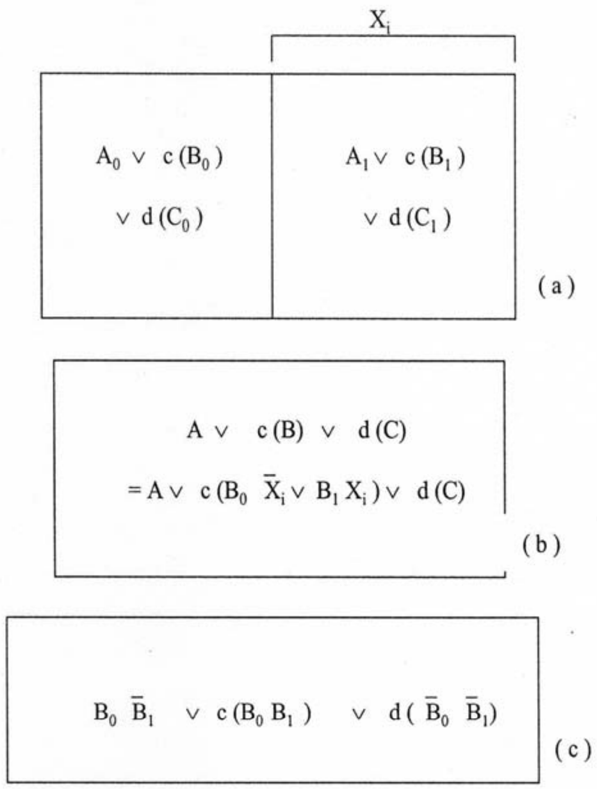

X_i
$A_0 \vee c(B_0) \vee d(C_0)$
$A_1 \vee c(B_1) \vee d(C_1)$
(a)
$A \vee c(B) \vee d(C)$
$= A \vee c(B_0 \bar{X}_i \vee B_1 X_i) \vee d(C)$
(b)
$B_0 \bar{B}_1 \vee c(B_0 B_1) \vee d(\bar{B}_0 \bar{B}_1)$
(c)

Fig. 1. Generation of a cell entry in the X_i VEKM in terms of cell entry/entries in the X_{i+1} VEKM : (a) pertinent cells in the X_{i+1} VEKM when X_i is a map variable, (b) pertinent cell in the X_{i+1} VEKM when X_i is an entered variable, (c) corresponding cell in the X_i VEKM.

4. An illustrative example

Let us apply the present method in terms of a VEKM representation to find the subsumptive general solution of an equation of the form $f(X_1, X_2, X_3) = 0$, where f is a Boolean function $f = \mathbf{B}^3 \rightarrow \mathbf{B}$ where $\mathbf{B}$ is the Boolean carrier of $2^{**}(2^2) = 2^4 = 16$ elements constructed as all binary functions of a and b. These 16 elements have a partial order among themselves according to the inclusion ($\leq$) operator, and constitute a 4-dimensional hypercube representing a complemented distributive lattice as shown in Fig. 2. Note that the input space of f consists of $16^3 = 4096$ combinations of **X**, but f is uniquely defined by the values assigned to it on only 8 combinations of **X**, namely those belonging to $\{0,1\}^3$. Let f be given by the formula

$$f = b X_1 \vee b \overline{X}_2 X_3 \vee \overline{b} \overline{X}_1 X_2 \vee \overline{a} \overline{X}_2 \overline{X}_3 \vee \overline{a} X_1 \overline{X}_2 \vee \overline{a} \overline{X}_1 X_2 \vee a \overline{b} X_2 X_3. \qquad (4.1)$$

This equation has been solved repeatedly by Brown [1], with his most simplified solution being via his Marquand-diagram procedure. The first step of this procedure is to expand f not only with respect to the 3 variables X_1, X_2, and X_3 but further with respect to the "constants" a and b thereby producing a 5-variable 32-cell Marquand diagram. This diagram is repeatedly folded to obtain Marquand diagrams for the eliminants $f_3 = f$, f_2, f_1 and f_0. For $i = 3, 2, 1$ diagram entries for s_i and t_i are produced in terms of diagram entries for f_i in a cellwise fashion. The functions s_i and t_i ($i = 3, 2, 1$) in Marquand-diagram form are not readily amenable to simplification or minimization.

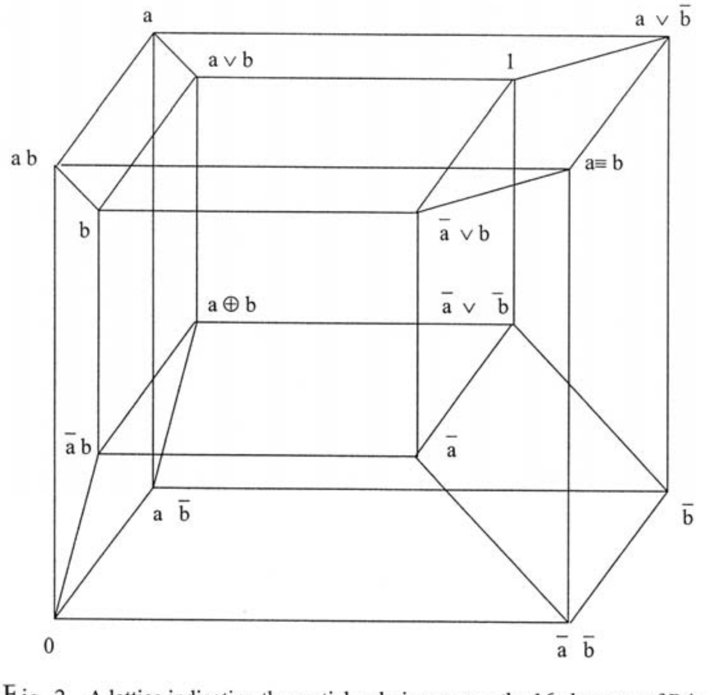
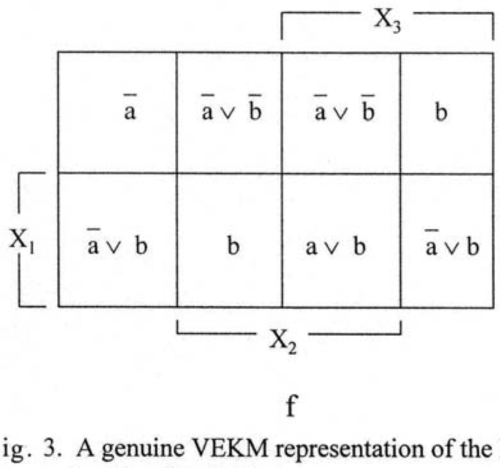


Fig. 2. A lattice indicating the partial ordering among the 16 elements of B in the example.

In our present procedure, however, we expand f only with respect to the true variables X_1, X_2, and X_3 , thereby representing f by the 8-cell VEKM in Fig. 3 which has X_1, X_2, and X_3 as map variables and has a and b as entered "variables". Since a and b are actually constants, this VEKM is a natural map for f. We switch the entries of this map into c entries and consider it a map for a fictitious variable X_4 as shown in Fig. 4(a). This new map is the first in a series of generator VEKMs X_4, X_3, X_2 and X_1 given in Figs. 4(b)-4(d). The entries of a VEKM X_i, i = 3, 2, 1 are deduced from the can't-happen entries of the predecessor or generator X_{i+1} VEKM according to the transformation from (a) to (c) in Fig. 1. Now, the VEKM for every s_i (t_i), i = 3, 2, 1, is obtained from that of the corresponding X_i with d set to 0(1) and c converted into d. The s_i and t_i VEKMs are equivalent to those obtained for the current example by Rushdi [5], and can be used to construct minimal sum-of-products expressions via the procedure of Rushdi and Al-Yahya [8]. Finally, the subsumptive general solution for f = 0 is obtained as

$$\begin{array}{rcccl} & & \bar{a}\, b & = & 0 \\ & & 0 \;\le & X_1 & \le\; \bar{b} \\ & & \bar{a} X_1 \;\le & X_2 & \le\; b \vee X_1 \\ \bar{a}\, \bar{X}_1 & \le & X_3 & \le & \bar{a} \vee \bar{b}\, \bar{X}_2 \vee b X_2 \,. \end{array} \tag{4.2}$$

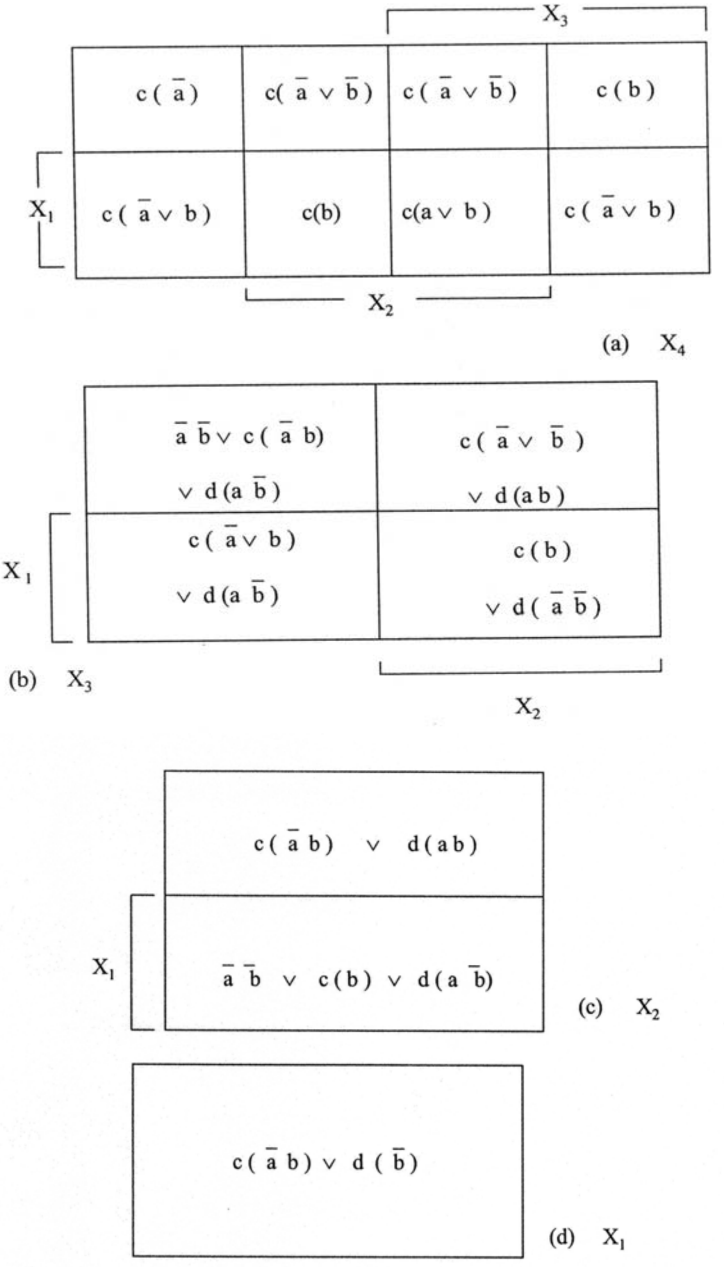


$\bar{a}$	$\bar{a} \vee \bar{b}$	$\bar{a} \vee \bar{b}$	b
$\bar{a} \vee b$	b	$a \vee b$	$\bar{a} \vee b$

Fig. 3. A genuine VEKM representation of the Boolean function f in (4.1).

The solution (4.2) is in its most simplified form, as verified from earlier work by Brown [1], Rushdi [5], and Rudeanu [10]. For comparison purposes, Fig. 5 shows the VEKMs of Fig. 4 expanded in CKM form, with the original VEKM boundaries stressed as thick bold lines. The initial map of the fictitious

X_4 variable is the CKM for f in (4.1) with 1's replaced by c's. The sequence of maps for X_3, X_2, X_1 in Fig. 5 is directly obtained via the rules in Table 2.



			X_3	X_3
	$c(\bar{a})$	$c(\bar{a} \vee \bar{b})$	$c(\bar{a} \vee \bar{b})$	$c(b)$
X_1	$c(\bar{a} \vee b)$	$c(b)$	$c(a \vee b)$	$c(\bar{a} \vee b)$
		X_2	X_2	

(a) X_4

	$\bar{a}\,\bar{b} \vee c(\bar{a}\,b)$ $\vee\, d(a\,\bar{b})$	$c(\bar{a} \vee \bar{b})$ $\vee\, d(ab)$
X_1	$c(\bar{a} \vee b)$ $\vee\, d(a\,\bar{b})$	$c(b)$ $\vee\, d(\bar{a}\,\bar{b})$
		X_2

(b) X_3

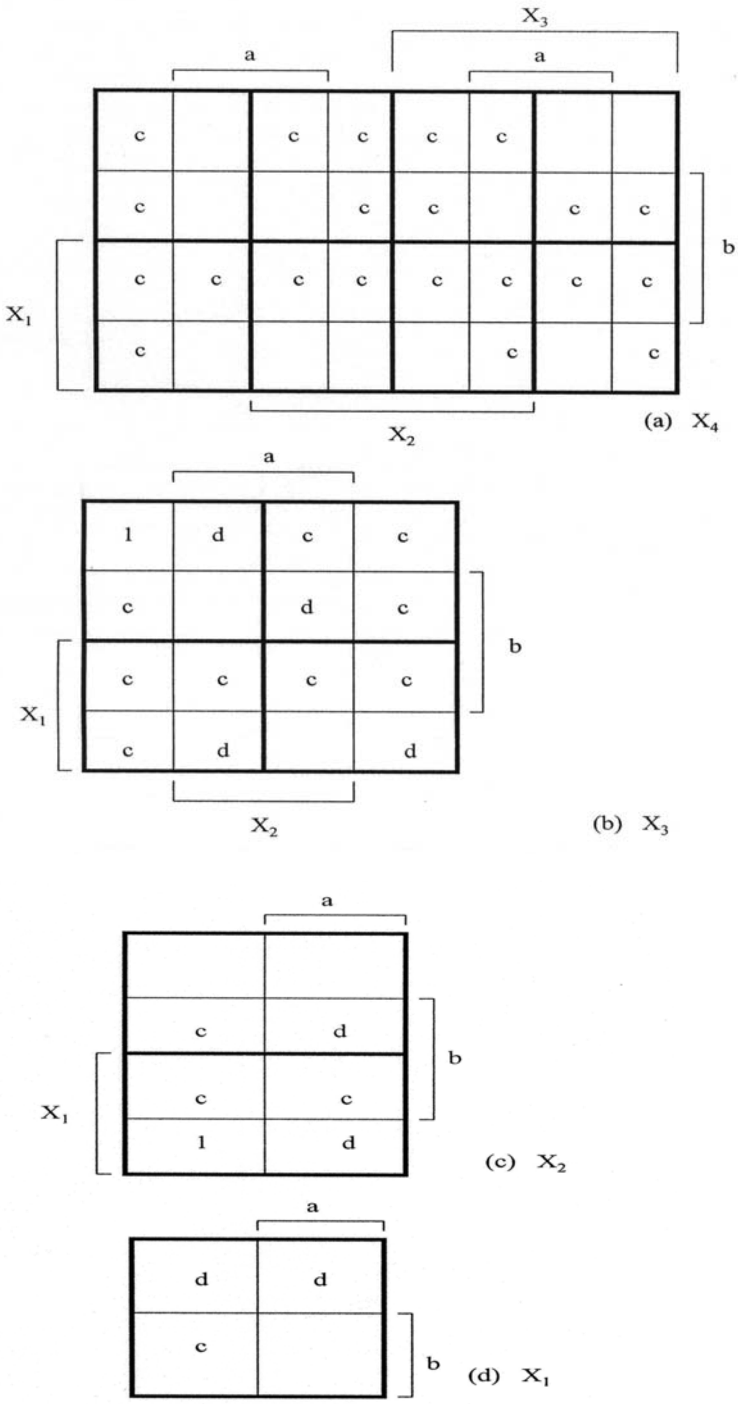
	$c(\bar{a}\,b) \vee d(ab)$
X_1	$\bar{a}\,\bar{b} \vee c(b) \vee d(a\,\bar{b})$

(c) X_2

$c(\bar{a}\,b) \vee d(\bar{b})$

(d) X_1

FIG. 4. VEKMs used in the construction of a subsumptive general solution in the example.



c		c	c	c	c		
c			c	c		c	c
c	c	c	c	c	c	c	c
c					c		c

(a) X$_4$

1	d	c	c
c		d	c
c	c	c	c
c	d		d

(b) X$_3$

c	d
c	c
1	d

(c) X$_2$

d	d
c	

(d) X$_1$

Fig. 5. The VEKMs in figure 3 expanded into CKM forms.

It is interesting to note that the consistency conditions $\bar{a}\, b = 0$ restricts the input space of **X** from $16^3 = 4096$ combinations to only $8^3 = 512$ combinations, since the original lattice of B of 16 elements (Fig. 2) collapses to one of 8 elements only (Fig. 6). In fact, the 4-dimensional hypercube in Fig. 2 loses one of its dimensions due to the cancellation of its atom $\bar{a}\, b$, and hence reduces to a 3-dimensional hypercube, or simply a cube. The general solution of (4.2) can now be used to generate all particular solutions with the aid of the partial order in Fig 6.

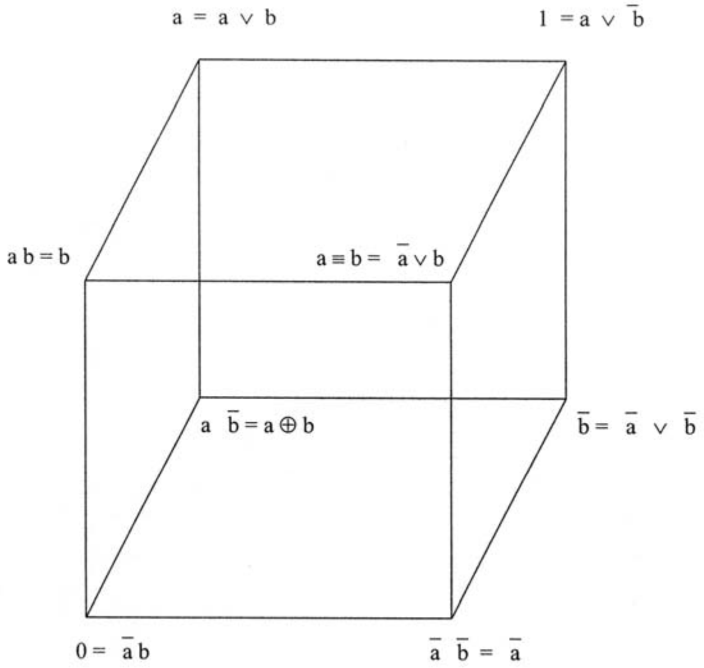

$a = a \vee b$
$1 = a \vee \bar{b}$
$a\, b = b$
$a \equiv b = \bar{a} \vee b$
$a\, \bar{b} = a \oplus b$
$\bar{b} = \bar{a} \vee \bar{b}$
$0 = \bar{a}\, b$
$\bar{a}\, \bar{b} = \bar{a}$

Fig. 6. The lattice in figure 2 when collapsed under the condition $\bar{a}\, b = 0$.

Table 3 explicitly lists all the particular solutions of the equation f = 0. These solutions are all the valid solutions (and nothing else) produced individually without any kind of overlapping or repetition. For each of these solutions f can be shown to equal $\bar{a}$ b which is zero according to the consistency condition. However, Table 3 is not the recommended form for a solution representation since it details a large number of solutions and obscures regularities in their form.

Table 3: Particular solutions equivalent to (4.2).

X_1	X_2	X_3	X_1	X_2	X_3
0	0	$\bar{a}$	$\bar{a}$	$\bar{a} \vee b$	1
0	0	$\bar{b}$	$\bar{b}$	$\bar{a}$	0
0	b	$\bar{a}$	$\bar{b}$	$\bar{a}$	$\bar{a}$
0	b	$\bar{b}$	$\bar{b}$	$\bar{a}$	$a\bar{b}$
0	b	$\bar{a} \vee b$	$\bar{b}$	$\bar{a}$	$\bar{b}$
0	b	1	$\bar{b}$	$\bar{b}$	0
$a\bar{b}$	0	$\bar{a}$	$\bar{b}$	$\bar{b}$	$\bar{a}$
$a\bar{b}$	0	$\bar{b}$	$\bar{b}$	$\bar{a} \vee b$	0
$a\bar{b}$	b	$\bar{a}$	$\bar{b}$	$\bar{a} \vee b$	$\bar{a}$
$a\bar{b}$	b	$\bar{b}$	$\bar{b}$	$\bar{a} \vee b$	$a\bar{b}$
$a\bar{b}$	b	$\bar{a} \vee b$	$\bar{b}$	$\bar{a} \vee b$	$\bar{b}$
$a\bar{b}$	b	1	$\bar{b}$	$\bar{a} \vee b$	b
$a\bar{b}$	$a\bar{b}$	$\bar{a}$	$\bar{b}$	$\bar{a} \vee b$	$\bar{a} \vee b$
$a\bar{b}$	a	$\bar{a}$	$\bar{b}$	$\bar{a} \vee b$	a
$a\bar{b}$	a	$\bar{a} \vee b$	$\bar{b}$	$\bar{a} \vee b$	1
$\bar{a}$	$\bar{a}$	0	$\bar{b}$	1	0
$\bar{a}$	$\bar{a}$	$\bar{a}$	$\bar{b}$	1	$\bar{a}$
$\bar{a}$	$\bar{a}$	$a\bar{b}$	$\bar{b}$	1	b
$\bar{a}$	$\bar{a}$	$\bar{b}$	$\bar{b}$	1	$\bar{a} \vee b$
$\bar{a}$	$\bar{a} \vee b$	0			
$\bar{a}$	$\bar{a} \vee b$	$\bar{a}$			
$\bar{a}$	$\bar{a} \vee b$	$a\bar{b}$			
$\bar{a}$	$\bar{a} \vee b$	$\bar{b}$			
$\bar{a}$	$\bar{a} \vee b$	b			
$\bar{a}$	$\bar{a} \vee b$	$\bar{a} \vee b$			
$\bar{a}$	$\bar{a} \vee b$	a			

5. Conclusions

This paper presents an efficient manual method for obtaining the most compact form of the subsumptive general solution of a system of Boolean equations. The method is an effective combination of mapping and algebraic methods (through its use of VEKM representations for Boolean functions and variables) and employs a minimum number of constructions (through its artificial distinction between don't-care and can't-happen conditions). Throughout its work, the method keeps track only of a *single-variable* representation that can have positively-asserted, don't-care, can't-happen or negatively-asserted components that are functions of the constants of the pertinent Boolean carrier. This single-variable representation leads to an immediate construction of the minimal sum-of-product expressions for the lower and upper bound on the present variable as well as to a generation of the next variable representation. In addition to being highly economic, the present method is not restricted to two-valued Boolean equations and is naturally suitable for 'big' Boolean algebras.

The concepts and method developed herein can be utilized in various application areas of Boolean equations [1,2,9,11,12]. In particular, an automated version of the present Boolean-equation solver can be applied in the simulation of gate-level logic. However, such an application must handle the incompatibility between the lattice structure of 'big' Boolean algebras, which are only partially ordered, and multi-valued logics, which are totally ordered [13]. The ideas expressed herein can also be incorporated in the automated solution of large systems of Boolean equations [11,14]. They can also be extended to handle quadratic Boolean equations [15], Boolean ring equations [9,16], and Boolean differential equations [17].

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الحل السريع للمعادلات البولانية باستخدام خرائط كارنوه متغيرة المحتويات

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المستخلص : يتم تقديم طريقة جديدة للحصول على حل عام احتوائي ملموم لنظام من المعادلات البولانية. وتعتمد الطريقة على استعمال خريطة كارنوه متغيرة المحتويات (خ ك غ ح) لتحقيق الحذف التتابعي من خلال الطي المتتابع للخريطة. كذلك تفتعل الطريقة تمييزًا مصطنعًا بين اشتراطات انعدام الأهمية واشتراطات امتناع الحدوث. ومن ثم فإنها تتمتع بكفاءة عالية حيث إنها تتطلب إنشاء خرائط تقل في العدد بوضوح كما تصغر في الحجم بوضوح عن تلك التي تتطلبها الطرائق التقليدية. وفضلاً عن ذلك، يمكن تطبيق الطريقة على المعادلات البولانية العامة دون التقيد بالحالة ثنائية القيمة. إن تفصيلات الطريقة يتم تعليلها بصورة رصينة، كما يجري شرحها بعناية ومن ثم يُعطى بيان لها من خلال ثلاثة أمثلة توضيحية.